Quiver matrix model and topological partition function in six dimensions

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ABSTRACT: We consider a topological quiver matrix model which is expected to give a dual description of the instanton dynamics of topological $U(N)$ gauge theory on $D6$ branes. The model is a higher dimensional analogue of the ADHM matrix model that leads to Nekrasov's partition function. The fixed points of the toric action on the moduli space are labeled by colored plane partitions. Assuming the localization theorem, we compute the partition function as an equivariant index. It turns out that the partition function does not depend on the vacuum expectation values of Higgs fields that break $U(N)$ symmetry to $U(1)^N$ at low energy. We conjecture a general formula of the partition function, which reduces to a power of the MacMahon function, if we impose the Calabi-Yau condition. For non Calabi-Yau case we prove the conjecture up to the third order in the instanton expansion.

Contents

1 Introduction

During the recent developments in the non-perturbative dynamics of supersymmetric gauge/string theories, we have witnessed many examples of the topological partition function which are exactly computable. They arise from the enumerative problems and are defined as the generating functions of instanton or BPS state counting. Thus they carry useful information for testing various dualities in supersymmetric theories, such as mirror symmetry, electro-magnetic duality and gauge/string correspondence. One of the important mathematical ideas in these computations is the equivariant localization theorem and it has revealed a close relation to the combinatorics. We use basic combinatorial tools in representation theory, such as the partition (the Young diagram), the plane partition, the Schur function and the Macdonald function. For example, N-tuple of the Young diagrams or the colored partition appears in Nekrasov's computation of Seiberg-Witten prepotential [1–3]. We use the (skew) Schur function to write down the topological vertex [4, 5], which gives a building block of topological string amplitudes on toric Calabi-Yau threefolds. It is also related to the plane partition [6]. The generating function of counting plane partitions is the MacMahon function, which is ubiquitous in topological gauge/string theory. For example, it appears in topological string amplitude on the conifold [7], the Gopakumar-Vafa invariants [8] and the Donaldson-Thomas theory [9–11]. Finally the Macdonald function, which is the most general class of the symmetric functions, was employed to construct a refinement of the topological vertex [12–15].

It is quite interesting that the topological partition function often takes the plethystic form.[1] Namely there exists a function $\mathcal{F}(t_1, t_2, \cdots)$ and the partition function is written as the plethystic exponential;

$$Z_{\text{top}}(t_1, t_2, \cdots) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \mathcal{F}(t_1^k, t_2^k, \cdots)\right), \tag{1.1}$$

where we have denoted parameters of the theory collectively as $(t_1, t_2, \cdots)$. This implies that the partition function has an infinite product (Euler product) form. Topological string amplitudes in the Gopakumar-Vafa form are basic examples. Furthermore the fact that Nekrasov's partition function allows an expansion in the plethystic form is crucial to identify it as topological string amplitudes or their refined version [12, 15]. The MacMahon function which is a basic partition function in the Donaldson-Thomas theory has also the plethystic form;

$$M(t) := \prod_{n=1}^{\infty} (1 - t^n)^{-n} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{(t^{\frac{k}{2}} - t^{-\frac{k}{2}})^2}\right). \tag{1.2}$$

It is an interesting challenge to uncover a possible mathematical and/or physical origin of the plethystic exponential in general.

In this paper we consider a topological quiver matrix model which is expected to describe low energy and instanton dynamics of the topological gauge theory on $D6$ branes [17, 18]. The model is a six dimensional analogue of the ADHM matrix model derived from low energy effective theory of $D4$-$D0$ system [19–22]. The instanton partition function for the ADHM matrix model is nothing but Nekrasov's partition function which is related to the Seiberg-Witten prepotential and topological string. The fixed points of the toric action on the moduli space of the ADHM matrix model are labeled by colored partitions [23]. Hence the localization theorem tells us that the partition function can be computed as a summation over colored partitions. We consider a similar instanton partition function for the six dimensional quiver matrix model. To compute the partition function based on the localization theorem, which we assume throughout the paper, we introduce T^3 action $(z_1, z_2, z_3) \rightarrow (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2, e^{i\epsilon_3} z_3)$ on $\mathbb{C}^3$, which may be regarded as the Ω background of Nekrasov. We also consider the action of the maximal torus $\mathrm{U}(1)^N$ of the gauge group $\mathrm{U}(N)$, where the rank N refers to the number of $D6$ branes. These toric actions induce the action on the moduli space of the topological quiver matrix model and the fixed points are labeled by N-tuples of plane partitions (3d Young diagrams), or colored plane partitions [18]. The partition function is defined as an equivariant index and thus a rational function of equivariant parameters $q_i = e^{i\epsilon_i}$ from T^3 action and $e_\alpha = e^{ia_\alpha}$ from the maximal torus. Physically a_α are the vacuum expectation values of Higgs fields. We call the condition $\mathbf{q} := \sqrt{q_1 q_2 q_3} = 1$ Calabi-Yau condition, which can be compared with the anti-self-duality in four dimensions. When the Calabi-Yau condition is imposed, the weight or the measure at any fixed point is ± 1 and consequently the partition function of

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[1]The plethystic exponential also appears in the problem of counting gauge invariant operators in quiver gauge theories. See for example [16].

U(N) theory reduces to the N-th power of the MacMahon function. Hence the partition function is independent of both q_i and e_α.

In non Calabi-Yau case the weight at each fixed point becomes rather complicated expression and the partition function does depend on the equivariant parameters q_i. However, we have found that even in this case, the instanton partition function is still independent of e_α for lower instanton numbers. We believe this is quite surprising. Based on explicit computations for lower rank N and instanton number k, we propose the following formula of the topological partition function;

$$Z_{6D}^{U(N)}(q_i; \Lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} F_N(q_1^n, q_2^n, q_3^n; \Lambda^n)\right), \tag{1.3}$$

where

$$F_N := -\frac{\widetilde{\Lambda}(1 + \mathbf{q}^2 + \mathbf{q}^4 + \cdots + \mathbf{q}^{2N-2})}{(1 - \widetilde{\Lambda})(1 - \mathbf{q}^{2N}\widetilde{\Lambda})} \frac{(1 - q_1 q_2)(1 - q_2 q_3)(1 - q_3 q_1)}{(1 - q_1)(1 - q_2)(1 - q_3)}, \tag{1.4}$$

and $\widetilde{\Lambda} := (-\mathbf{q})^{-N}\Lambda$ is a renormalized parameter of the instanton expansion parameter Λ. Note that when $\mathbf{q} = 1$, q_i dependence of the partition function disappears completely and we have $Z_{CY3}^{U(N)} = M(\widetilde{\Lambda})^N$. If we expand the first factor of F_N in $\widetilde{\Lambda}$, the coefficients are $\mathbf{q}$-integers. Thus the above formula may have a certain interpretation of $\mathbf{q}$-deformation.

Mathematically the topological partition function we compute in this paper has a natural meaning in K theory. The K theoretic version of Nekrasov's partition function is physically regarded as a five dimensional lift and it is the K theoretic version which we can identity with topological string amplitudes. Thus we expect that our partition function has a seven dimensional interpretation, if we combine it with the perturbative contributions from Kaluza-Klein modes. In fact the partition function (1.3) for abelian theory ($N = 1$) was conjectured in [24] together with a curious relation to M theory partition function. It is possible that the simplicity of the topological partition function we proposed above originates from the maximally supersymmetry of Yang-Mills gauge theory. In fact we have encountered before a similar example in five dimensional U(1) gauge theory with an adjoint hypermultiplet [15, 25–27]. In this case Nekrasov's partition function takes the following form of the plethystic exponential;

$$Z_{5D}^{adj}(q_i, Q; \Lambda) = \sum_{\lambda} \Lambda^{|\lambda|} \prod_{s \in \lambda} \frac{1 - Q q_1^{-a(s)} q_2^{\ell(s)+1}}{1 - q_1^{-a(s)} q_2^{\ell(s)+1}} \frac{1 - Q q_1^{a(s)+1} q_2^{-\ell(s)}}{1 - q_1^{a(s)+1} q_2^{-\ell(s)}}$$

$$= \exp\left\{\sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1 - Q^n \Lambda^n} \frac{(1 - Q^n q_1^n)(1 - Q^n q_2^n)}{(1 - q_1^n)(1 - q_2^n)}\right\}. \tag{1.5}$$

The left hand side of (1.5) is a summation over partitions λ. The integers $a(s)$ and $\ell(s)$ in the product are the arm length and the leg length that are defined by the corresponding Young diagram. The mass m of the adjoint hypermultiplet defines the parameter $Q = e^{-m}$ of the mass deformation. It is tempting to compare the parameter $\mathbf{q}$ in (1.4) with Q in (1.5).

The topological quiver matrix model in this paper is a $0 + 1$-dimensional "world-line" theory on $D0$ branes. However, we should emphasize that if the Donaldson-Thomas theory

is formulated as a topological gauge theory on $D6$ branes, the effect of $D2$-branes on $D6$ cannot be negligible. If we want to regard the topological quiver matrix model as a dual description of the Donaldson-Thomas theory, it has to accommodate $D2$ branes. For example the contribution of "0-2" string should appear as a multiplicative factor to the partition function (1.3). Recall that in [6] the enumeration of plane partitions led to the generating function $Z_{\lambda\mu\nu}(u) = M(u)C_{\lambda\mu\nu}(u)$, where three partitions (Young diagrams) λ, μ, ν define asymptotic conditions on plane partitions and $u := e^{-g_s}$ is related to the string coupling g_s. The generating function is given by the topological vertex $C_{\lambda\mu\nu}(u)$ and the MacMahon function appears as a normalization factor. From this viewpoint what we have proposed above is an extension of the MacMahon factor to non Calabi-Yau case. Thus the issue is closely related to the problem of extending the topological vertex to toric Kähler threefolds. In any case incorporating the effect of $D2$ branes to the quiver matrix model is beyond the scope of the present paper. We want to address this issue in future.

The paper is organized as follows; in section 2 we review the construction of topological quiver matrix model following [17, 18, 28, 29]. We clarify the relation of the stability condition and the vanishing theorem, which was not emphasized before in [17, 18]. In section 3 we introduce the toric action on the moduli space of the topological matrix model. The fixed points are isolated and they are classified by N-tuples of plane partitions (colored plane partitions). Hence we can compute the partition function which is defined as an equivariant index by summing up the contribution at each colored plane partition. When the Calabi-Yau condition is imposed, the partition function reduces to a power of MacMahon function. Section 4 is the main part of the paper. We compute the partition function for non Calabi-Yau case and prove our conjecture (1.3) up to instanton number three. We first look at possible poles of the partition function in e_α and show that all the residues vanish. Hence the partition function is independent of e_α and we compute it by taking appropriate limit (the decoupling limit). It turns out that the identities among **q**-integers derived from the **q** binomial theorem reduce the conjecture for U(N) theory to that for U(1) theory. We can check the conjecture for U(1) theory by direct computation. In appendix A we give a brief review of the ADHM matrix model and the relation to Nekrasov's partition function. In appendix B we check the well-definedness of our partition function.

2 Topological quiver matrix model

Let us consider a quiver matrix model for topological gauge theory on $D6$ brane [17, 18]. This is an analogue of the ADHM matrix model. As in the ADHM construction we introduce two vector spaces V and W of complex dimensions $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} W = N$. From the perspective of the gauge theory on the world volume of $D6$ brane, k is the number of $D0$ branes (the instanton number) and N is the number of $D6$ branes (the rank). The basic fields are the matrices

$$B_1, B_2, B_3, \varphi \in \text{Hom}(V, V), \qquad I \in \text{Hom}(W, V), \tag{2.1}$$

where (B_1, B_2, B_3, φ) come from "0-0" string and I comes from "6-0" string. Compared with the ADHM date for four dimensional gauge theory, the model does not have

$J \in \text{Hom } (V, W)$, or "0-6" string.[2] Instead, we have a matrix $\varphi \in \text{Hom } (V, V)$. This additional field comes from a reduction of a topological theory in eight dimensions [31], which originates from the ten dimensional super Yang-Mills theory. We can show the vanishing theorem which implies that $\varphi = 0$ on the moduli space [31]. Hence, classically φ is irrelevant to the moduli problem. However, the presence of φ is crucial for imposing the constraint (2.4) to be introduced below. We consider the following equations of motion[3]

$$\mathcal{E}_F := [B_i, B_j] + \epsilon_{ijk}[B_k^\dagger, \varphi] = 0, \tag{2.2}$$

$$\mathcal{E}_D(\zeta) := \sum_{i=1}^{3}[B_i, B_i^\dagger] + [\varphi, \varphi^\dagger] + II^\dagger - \zeta \cdot \mathbf{1}_{k \times k} = 0, \quad (\zeta > 0), \tag{2.3}$$

$$\mathcal{E}_B := I^\dagger \varphi = 0 . \tag{2.4}$$

These are the gauge fixing conditions or the constraints in our topological matrix model with gauge symmetry U(k). Among them (2.2) gives three F-tern (holomorphic) conditions and (2.3) is the (real) D-term condition which is responsible for the stability. There is no counter part of (2.4) in the ADHM equation and it may be interesting to clarify its implication. Since φ describes the normal direction to the world volume of $D6$ branes, (2.4) means that the "6-0" string I is orthogonal to the normal direction [17]. This implies that $D0$ branes are forced to be bound to $D6$ branes. The reason why we should impose the constraint (2.4) might be related to the fact that $D6$-$D0$ system cannot make a BPS bound state without an appropriate flux along the $D6$ branes [32]. In any case it is important to further clarify a possible explanation from the viewpoint of the BPS states.

One can construct a topological matrix model following the prescription of [28, 29]. This was achieved in [17, 18]. Since we have the constraints (2.2)–(2.4), the moduli space of the topological theory is identified with

$$\mathcal{M}_{\text{TQM}} := \{(B_i, \varphi, I) \mid \mathcal{E}_F = \mathcal{E}_D(\zeta) = \mathcal{E}_B = 0\}/\text{U}(k) . \tag{2.5}$$

Let us count the degrees of freedom. Since B_i, φ and I are complex matrices, there are $8k^2 + 2Nk$ degrees of freedom. The constraints impose $6k^2 + k^2 + 2Nk$ relations. Finally we have U(k) gauge symmetry. Hence the formal dimension of the moduli space $\mathcal{M}_{\text{TQM}}$ vanishes. This is certainly due to the fact that the origin of our theory is the ten dimensional super Yang-Mills theory which is maximally supersymmetric. However, this gives us a puzzle, since we naively expect $6Nk$ degrees of freedom for k $D0$ branes bound to N $D6$ branes, which means the (complex) dimensions of the moduli space are $3Nk$. We suspect the following fact is related to this issue. According to the general theory of topological matrix model [28, 29], we are computing the Euler character of the anti-ghost bundle on the moduli space $\mathcal{M}_{\text{TQM}}$. However, in the present case, the anti-ghost bundle only makes sense as a complex of bundles [17] and it is in fact different from the tangent bundle of $\mathcal{M}_{\text{TQM}}$ which is defined by the linearization of the constraints.

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[2]See [30] for an explanation from the viewpoint of T duality.

[3]See also [32–34] on the equations of ADHM type for $D0$-$D6$ and $D0$-$D8$ systems. In these papers a background B-field was introduced to obtain BPS bound states.

In the ADHM matrix model the moduli space of the type (2.5) comes from the hyperKähler quotient construction. It is well known that we have an equivalent definition of the moduli space by affine algebro-geometric quotient [3, 23], where we omit the D-term condition but impose the stability condition on the orbits. We also have to complexify the gauge group to $GL(k, \mathbb{C})$. Since it is not established at the moment that a similar equivalence holds in our higher dimensional generalization, we assume it and we take

$$\widetilde{\mathcal{M}}_{\mathrm{TQM}} := \{(B_i, \varphi, I) \mid \mathcal{E}_F = \mathcal{E}_B = 0, \text{ stability condition}\} // GL(k, \mathbb{C}) , \qquad (2.6)$$

as our definition of the moduli space in the following. In (2.6) $//$ means the affine algebro-geometric quotient where we only consider the orbits that satisfy the stability condition;

$$\text{There is no proper subspace } S \subsetneq V \quad \text{with} \quad B_i S \subset S, \quad \mathrm{Im}(I) \subset S . \qquad (2.7)$$

We now show that under the stability condition (2.7) we have a vanishing theorem that $\varphi = 0$, if $(B_i, \varphi, I) \in \widetilde{\mathcal{M}}_{\mathrm{TQM}}$. Firstly, we note that the F-term condition $\mathcal{E}_F = 0$ splits into two independent equations,

$$[B_i, B_j] = [B_k^\dagger, \varphi] = 0 . \qquad (2.8)$$

To see it, let $A_k := [B_k^\dagger, \varphi] \in \mathrm{Hom}\,(V, V)$. Then by (2.2) and the Jacobi identity, we have

$$\mathrm{Tr}\, A_k^\dagger A_k = \frac{1}{2}\epsilon_{ijk}\mathrm{Tr}\, [\varphi^\dagger, B_k][B_i, B_j] = \frac{1}{2}\epsilon_{ijk}\mathrm{Tr}\, \varphi^\dagger[[B_i, B_j], B_k] = 0 . \qquad (2.9)$$

Hence $A_k = 0$ and (2.8) holds. To prove the vanishing theorem it is enough to show that $\varphi^\dagger v = 0$ for any $v \in V$. By the stability condition the vector space V is generated by applying B_i's on $\mathrm{Im}(I)$. Hence any vector $v \in V$ can be written as $v = B_{i_1} B_{i_2} \cdots B_{i_n} I(w)$ by choosing an appropriate vector $w \in W$. Since $\varphi^\dagger$ and B_i's commute by (2.8), $\varphi^\dagger v = B_{i_1} B_{i_2} \cdots B_{i_n} \varphi^\dagger I(w) = 0$, where the last equality follows from $\mathcal{E}_B = 0$. This completes the proof of the vanishing theorem. Consequently the matrix φ decouples and the moduli space is actually

$$\widetilde{\mathcal{M}}_{\mathrm{TQM}} = \{(B_i, I) \mid [B_i, B_j] = 0, \text{ stability condition}\} // GL(k, \mathbb{C}) . \qquad (2.10)$$

Note that the matrix I only concerns the stability condition.

It follows from (2.10) that when $N = 1$, we can identify $\widetilde{\mathcal{M}}_{\mathrm{TQM}}$ with the Hilbert scheme of k points in $\mathbb{C}^3$;

$$\mathrm{Hilb}^k(\mathbb{C}^3) = \{\mathcal{I} \subset \mathbb{C}[z_1, z_2, z_3] \mid \dim(\mathbb{C}[z_1, z_2, z_3]/\mathcal{I}) = k\} , \qquad (2.11)$$

where $\mathcal{I}$ denotes an ideal in the polynomial ring $\mathbb{C}[z_1, z_2, z_3]$. We note that $\mathbb{C}[B_1, B_2, B_3] \simeq \mathbb{C}[z_1, z_2, z_3]$, since $[B_i, B_j] = 0$. For any ideal $\mathcal{I} \in \mathrm{Hilb}^k(\mathbb{C}^3)$, let $V = \mathbb{C}[z_1, z_2, z_3]/\mathcal{I}$. We define $B_i \in \mathrm{Hom}\,(V, V)$ by the multiplication of z_i modulo $\mathcal{I}$. When $N = 1$, I is defined by giving an element $I(1) \in V$. We take $I(1) = 1$ modulo $\mathcal{I}$. Then clearly $[B_i, B_j] = 0$ and it is easy to see that the stability condition is satisfied. Conversely, for any element $(B_i, I) \in \widetilde{\mathcal{M}}_{\mathrm{TQM}}$, we define a map $\mu : \mathbb{C}[z_1, z_2, z_3] \to V$ by $\mu(f(z_1, z_2, z_3)) := f(B_1, B_2, B_3) \cdot I(1)$, which is well-defined thanks to $[B_i, B_j] = 0$. The stability condition implies that μ is surjective. Hence, if we define an ideal in $\mathbb{C}[z_1, z_2, z_3]$ by $\mathcal{I} := \mathrm{Ker}\,\mu$, then $\mathbb{C}[z_1, z_2, z_3]/\mathcal{I} \simeq V$. Since $\dim_\mathbb{C} V = k$, we have $\mathcal{I} \in \mathrm{Hilb}^k(\mathbb{C}^3)$.

We can write down the deformation complex associated with the moduli space (2.6) by the standard manner;

$$
\begin{array}{ccccc}
& & \oplus_{k=1}^3 \mathrm{Hom}\ (V,V)_k & & \\
& & \oplus & & \oplus_{i,j=1}^3 \mathrm{Hom}\ (V,V)_{[ij]} \\
\mathrm{Hom}\ (V,V) & \xrightarrow{\ \sigma\ } & \mathrm{Hom}\ (V,V) & \xrightarrow{\ \tau\ } & \oplus \\
& & \oplus & & \mathrm{Hom}\ (V,W) \\
& & \mathrm{Hom}\ (W,V) & &
\end{array}
\tag{2.12}
$$

where the first term corresponds to the degrees of freedom of infinitesimal gauge transformation, the middle term parametrizes the tangent space of $\widetilde{\mathcal{M}}_{\mathrm{TQM}}$ and the last term comes from the linearization of the constraints (2.2) and (2.4). At a point $(B_i, \varphi, I) \in \widetilde{\mathcal{M}}_{\mathrm{TQM}}$ the maps σ and τ are defined by

$$
\sigma(\phi) := \delta_\phi(B_i, \varphi, I) = ([\phi, B_i], [\phi, \varphi], \phi I), \tag{2.13}
$$

$$
\tau((\delta B_i, \delta \varphi, \delta I)) := \left([\delta B_i, B_j] + [B_i, \delta B_j] + \epsilon_{ijk}([\delta B_k^\dagger, \varphi] + [B_k^\dagger, \delta \varphi]), \delta I^\dagger \varphi + I^\dagger \delta \varphi \right). \tag{2.14}
$$

Note that the gauge invariance of the constraints implies $\tau \circ \sigma = 0$.

3 Instanton partition function

Generalizing the computation of Nekrasov's partition function as the topological partition function of the ADHM matrix model, which is reviewed in appendix A, we want to compute the partition function of our quiver matrix model. By introducing the toric action on the moduli space and applying the localization theorem the partition function is computed as an equivariant index. We consider two toric actions on the moduli space $\widetilde{\mathcal{M}}_{\mathrm{TQM}}$. The first one comes from the canonical T^3 action $(z_1, z_2, z_3) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2, e^{i\epsilon_3} z_3)$ on $\mathbb{C}^3$, which is an example of the Ω background of Nekrasov. The second one is induced from the action of the maximal torus $\mathrm{U}(1)^N$ of the global gauge group $\mathrm{U}(N)$. Physically the parameters $a_\alpha, (\alpha = 1, \cdots, N)$ of the maximal torus correspond to the vacuum expectation values of the Higgs scalars or the distances of $D6$ branes. In the following we use the notations $q_i := e^{i\epsilon_i}, e_\alpha := e^{ia_\alpha}$. Since we have $GL(k, \mathbb{C})$ gauge symmetry, the condition of the fixed point is imposed up to gauge transformations. Hence, the conditions we have to solve are

$$
q_j B_j = g(q_i, \lambda) \cdot B_j \cdot g^{-1}(q_i, \lambda), \qquad (j = 1, 2, 3) \tag{3.1}
$$

$$
q_1 q_2 q_3 \varphi = g(q_i, \lambda) \cdot \varphi \cdot g^{-1}(q_i, \lambda), \tag{3.2}
$$

$$
I \cdot \lambda = g(q_i, \lambda) \cdot I, \tag{3.3}
$$

where $\lambda \in \mathrm{U}(1)^N$. Note that at each fixed point the conditions (3.1)–(3.3) define a homomorphism $g : T^3 \times \mathrm{U}(1)^N \to GL(k, \mathbb{C})$. By the homomorphism g we can regard the vector spaces W and V, which were originally $GL(k, \mathbb{C})$ modules, as $T^3 \times \mathrm{U}(1)^N$ modules. Through the matrix I the action of the maximal torus $\mathrm{U}(1)^N$ on W is translated into a $\mathrm{U}(1)^N$ action on V. It is helpful to keep these points in mind, when we compute the equivariant character of the deformation complex. In the following we will identify one dimensional $T^3 \times \mathrm{U}(1)^N$

modules with the equivariant parameters of the toric action. Namely q_i and e_α stand for the module where $T^3 \times U(1)^N$ acts as the multiplication of q_i and e_α, respectively. Hence a product of the equivariant parameters is regarded as a tensor product of one dimensional modules. Similarly q_i^{-1} and e_α^{-1} represent the dual modules and a sum of monomials in the equivariant parameters represents a direct sum of one dimensional modules.

We can classify the fixed points by generalizing the argument in [23]. The outcome is that they are labeled by N-tuples of plane partitions $\vec{\pi}$ (three dimensional Young diagrams), which we call colored plane partition in this paper. To be more precise, in non abelian case $N > 1$ we have to assume that the vacuum expectation values a_α are distinct each other. This means that the theory is in the Coulomb phase where the $U(N)$ gauge symmetry is completely broken. Let us take a basis $\{w_\alpha\}$ of W such that $U(1)^N$ acts by the multiplication of $e_\alpha^{-1} = e^{-ia_\alpha}$ on w_α. This is possible, since we have assumed that $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. Then we can show that

$$V = \oplus_{\alpha=1}^N V_\alpha, \qquad V_\alpha := \mathbb{C}[B_1, B_2, B_3] \cdot I(w_\alpha), \qquad (3.4)$$

where we allow that $V_\alpha = \{0\}$ for some α. Since $V = V_1 + V_2 + \cdots + V_N$ by the stability condition, it is enough to show that $V_\alpha \cap V_\beta = \{0\}$, if $\alpha \neq \beta$. Let $v \in V_\alpha \cap V_\beta$ and $g_\lambda := g(1, \lambda)$. Then we can write $v = B_{i_1} \cdots B_{i_n} I(w_\alpha) = B_{j_1} \cdots B_{j_m} I(w_\beta)$ and by (3.1) with $q_i = 1$ and (3.3), we have both $g_\lambda v = B_{i_1} \cdots B_{i_n} g_\lambda I(w_\alpha) = e_\alpha^{-1} v$ and $g_\lambda v = B_{j_1} \cdots B_{j_m} g_\lambda I(w_\beta) = e_\beta^{-1} v$. Hence $v = 0$, since $e_\alpha \neq e_\beta$. By the vanishing theorem the condition (3.2) is empty. To see the consequence of the remaining condition (3.1), we consider the decomposition $V_\alpha = \oplus_{i,j,k \in \mathbb{Z}} V_\alpha(i-1, j-1, k-1)$, where the eigenspace of $g_q := g(q_i, 1)$ is

$$V_\alpha(i-1, j-1, k-1) = \left\{ v \in V_\alpha \mid g_q v = q_1^{1-i} q_2^{1-j} q_3^{1-k} v \right\} . \qquad (3.5)$$

Then by the conditions of the fixed points, it is easy to see that $I(w_\alpha) \in V_\alpha(0, 0, 0)$ and that $B_1(V_\alpha(i, j, k)) \subset V_\alpha(i-1, j, k), B_2(V_\alpha(i, j, k)) \subset V_\alpha(i, j-1, k), B_3(V_\alpha(i, j, k)) \subset V_\alpha(i, j, k-1)$. Furthermore, as was shown in [18];

1. $V(i, j, k) = \{0\}$, if one of i, j, k is non-positive.

2. $\dim V(i, j, k) = 0$, or 1.

3. $\dim V(i, j, k) \geq \dim V(i+1, j, k)$ and similar inequalities for j and k.

For proofs of these facts, we refer to [18]. It is obvious that we can associate a plane partition π_α to the above decomposition data of V_α. Conversely, from an N-tuple of plane partitions $(\pi_1, \pi_2, \cdots \pi_N)$, one can construct a homomorphism $g : T^3 \times U(1)^N \to GL(k, \mathbb{C})$ that solves the conditions (3.1) and (3.3). Thus the fixed points of the toric action are isolated and they are labeled by colored plane partitions.

We can identify the plane partition π with the set $\{(i, j, k) \in \mathbb{N}^3 \mid k \leq h(i, j)\}$, where the height function $h(i, j) \in \mathbb{Z}_{\geq 0}$ satisfies $h(i, j) \geq h(i+1, j), h(i, j) \geq h(i, j+1)$. The size of the plane partition is defined by the volume of the corresponding set $|\pi| := \sum_{(i,j)} h(i, j)$. The size of the colored plane partition $\vec{\pi} = (\pi_1, \pi_2, \cdots, \pi_N)$ is defined by $|\vec{\pi}| := \sum_{\alpha=1}^N |\pi_\alpha|$.

By the localization theorem the partition function of our quiver matrix model is expressed as a summation over colored plane partitions

$$Z_{6D}^{U(N)}(q_i, e_\alpha; \Lambda) = \sum_{\vec{\pi}} \Lambda^{|\vec{\pi}|} N_{\vec{\pi}}(q_i, e_\alpha),$$ (3.6)

where Λ is the parameter of instanton expansion. As we will see shortly, the size of the colored plane partition $|\vec{\pi}|$ is identified with the instanton number k. The weight or the measure $N_{\vec{\pi}}(q_i, e_\alpha)$ at a fixed point $\vec{\pi}$ is a rational function of the equivariant parameters e_α and q_i. It physically represents the quantum fluctuation around each fixed point. To compute it at $\vec{\pi}$ we decompose V and W as $T^3 \times U(1)^N$ module as follows

$$W_{\vec{\pi}} = \sum_{\alpha=1}^{N} e_\alpha^{-1}, \qquad V_{\vec{\pi}} = \sum_{\alpha=1}^{N} e_\alpha^{-1} \left(\sum_{(i,j,k)\in\pi_\alpha} q_1^{1-i} q_2^{1-j} q_3^{1-k} \right).$$ (3.7)

The dual modules are

$$W_{\vec{\pi}}^* = \sum_{\alpha=1}^{N} e_\alpha, \qquad V_{\vec{\pi}}^* = \sum_{\alpha=1}^{N} e_\alpha \left(\sum_{(i,j,k)\in\pi_\alpha} q_1^{i-1} q_2^{j-1} q_3^{k-1} \right).$$ (3.8)

These are direct sum decompositions of V and W at $\vec{\pi}$ into one dimensional $T^3 \times U(1)^N$ modules or the characters of $T^3 \times U(1)^N$. Note that $\dim_{\mathbb{C}} W = N$ as it should be. Since $\dim_{\mathbb{C}} V = k$, we should have $|\vec{\pi}| = k$. From the toric action (3.1)–(3.3) we see the equivariant version of the deformation complex at $\vec{\pi}$ is

$$\begin{array}{ccccc}
 & & \text{Hom}\,(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes Q & & \\
 & & \oplus & & \text{Hom}\,(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes \Lambda^2 Q \\
\text{Hom}\,(V_{\vec{\pi}}, V_{\vec{\pi}}) & \xrightarrow{\sigma} & \text{Hom}\,(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes \Lambda^3 Q & \xrightarrow{\tau} & \oplus \\
 & & \oplus & & \text{Hom}\,(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes \Lambda^3 Q \\
 & & \text{Hom}\,(W_{\vec{\pi}}, V_{\vec{\pi}}) & &
\end{array},$$ (3.9)

where $Q = q_1 + q_2 + q_2$. Hence, the character of the deformation complex is

$$\chi_{\vec{\pi}} = V^* \otimes V \otimes (Q + \Lambda^3 Q) + W^* \otimes V - V^* \otimes V \otimes (1 + \Lambda^2 Q) - W \otimes V^* \otimes \Lambda^3 Q$$
$$= W^* \otimes V - W \otimes V^*(q_1 q_2 q_3) - V \otimes V^*(1 - q_1)(1 - q_2)(1 - q_3).$$ (3.10)

That is

$$\chi_{\vec{\pi}} = \sum_{\alpha,\beta=1}^{N} \frac{e_\alpha}{e_\beta} \left(\sum_{(i,j,k)\in\pi_\beta} q_1^{1-i} q_2^{1-j} q_3^{1-k} - \sum_{(r,s,t)\in\pi_\alpha} q_1^r q_2^s q_3^t - \sum_{\substack{(r,s,t)\in\pi_\alpha \\ (i,j,k)\in\pi_\beta}} q_1^{r-i} q_2^{s-j} q_3^{t-k} \prod_{\ell=1}^{3}(1 - q_\ell) \right).$$ (3.11)

We first note that in the character $\chi_{\vec{\pi}}$ the number of the terms with positive coefficient and those with negative coefficient coincide if we take the multiplicity into account. This is due

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to the fact that the formal dimension of the moduli space vanish and hence the character should vanish if we substitute $q_i = e_\alpha = 1$. Therefore we can write the character as

$$\chi_{\vec{\pi}}(q_i, e_\alpha) = \sum_{i=1}^{m} e^{w_i^{(+)}} - \sum_{i=1}^{m} e^{w_i^{(-)}}, \tag{3.12}$$

where $e^{w_i^{(\pm)}}$ are monomials in $q_i^{\pm}$ and $e_\alpha^{\pm}$. By the symmetry $\chi_{\vec{\pi}}(q_i, e_\alpha) = -q_1 q_2 q_3 \chi_{\vec{\pi}}(q_i^{-1}, e_\alpha^{-1})$, we can set $e^{w_i^{(-)}} = q_1 q_2 q_3 e^{-w_i^{(+)}}$. Hence if $e^{w_i^{(+)}} = e^{w_j^{(-)}}$ with $i \neq j$ then $e^{w_j^{(+)}} = e^{w_i^{(-)}}$. But $e^{w_i^{(+)}} \neq e^{w_i^{(-)}}$ in general, because if $e^{w_i^{(+)}} = e^{w_i^{(-)}}$ then $e^{w_i^{(+)}} = \sqrt{q_1 q_2 q_3}$. We will also show in appendix B that $e^{w_i^{(\pm)}} \neq (q_1 q_2 q_3)^n$ ($n \in \mathbb{Z}$). Then, according to the localization theorem the weight function is given by

$$N_{\vec{\pi}}(q_i, e_\alpha) = \prod_{i=1}^{m} \frac{\sinh w_i^{(-)}}{\sinh w_i^{(+)}}. \tag{3.13}$$

Compared with the computation in [18], the weight (3.13) computes the so-called K theoretic version of the partition function. For the ADHM matrix model the K theoretic version of Nekrasov's partition function corresponds to a five dimensional lift, where the relation to topological string amplitudes becomes more transparent [1, 35–38].

When we impose the Calabi-Yau condition $\mathbf{q} := \sqrt{q_1 q_2 q_3} = 1$, the character reduces to

$$\chi_{\vec{\pi}} = W^* \otimes V - W \otimes V^* + V \otimes V^*(q_1 + q_2 + q_3 - q_1^{-1} - q_2^{-1} - q_3^{-1}). \tag{3.14}$$

Since $W^*(e_\alpha) = W(e_\alpha^{-1})$ and $V^*(q_i, e_\alpha) = V(q_i^{-1}, e_\alpha^{-1})$, we see that under $q_i \to q_i^{-1}, e_\alpha \to e_\alpha^{-1}$, the sign of the character changes $\chi_{\vec{\pi}} \to -\chi_{\vec{\pi}}$. Therefore we can put $w_i^{(-)} = -w_i^{(+)}$ and hence

$$N_{\vec{\pi}}(q_i, e_\alpha) = (-1)^m. \tag{3.15}$$

Though the integer m may change, even if N and k are fixed, the parity of m and Nk agrees; $(-1)^m = (-1)^{Nk}$. Hence, the partition function is

$$Z_{\text{CY3}}^{\text{U}(N)}(q_i, e_\alpha; \Lambda) = \sum_{\vec{\pi}} \Lambda^{|\vec{\pi}|}(-1)^{N|\vec{\pi}|} = \prod_{\alpha=1}^{N} \sum_{\pi_\alpha} u^{|\pi_\alpha|} = M(u)^N, \tag{3.16}$$

where $u := (-1)^N \Lambda$ and $M(u)$ is the MacMahon function. This result was already obtained in [18]. Note that the argument of the MacMahon function is not the equivariant parameters of the toric action but the parameter of instanton expansion.[4] The fact that the weight of each fixed point is ± 1 reminds us of the topologically twisted $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [39].

---

[4]However, according to [24] it is possible to regard the parameter $\widetilde{\Lambda}$ as a part of Ω background of 11 dimensional supergravity, or M theory.

4 Computations in non Calabi-Yau case

In the last section we have seen that the partition function reduces to a power of the MacMahon function if we impose the Calabi-Yau condition. In particular, it is completely independent of both q_i and e_α. This is a remarkable difference from Nekrasov's partition function Z_{Nek}. When we impose the self-duality condition $\epsilon_1 + \epsilon_2 = 0$, Z_{Nek} is a function of $q := e^{-g_s} = q_1 = q_2^{-1}$ and a_α. The leading term of the genus expansion by g_s gives the Seiberg-Witten prepotential and the full expansion is identified with topological string amplitudes. For generic equivariant parameters q_1 and q_2, it is expected that Nekrasov's partition function gives a certain refinement of topological string amplitudes [12, 15, 40]. Thus it is interesting to see what happens to our instanton partition function, if we do not impose the Calabi-Yau condition.

For non Calabi-Yau case the weight function $N_{\vec{\pi}}(q_i, e_\alpha)$ no longer takes a simple form and is a rather complicated function. To obtain an idea on the structure of the partition function we made some explicit computations for lower rank and lower instanton number and found that the partition function is independent of e_α. In the following by examining the residues we will confirm this up to three instanton number for general N. Based on these explicit computations of several examples, we strongly believe that this property holds for higher instanton numbers and conjecture that the full partition function is given by

$$Z_{6D}^{\mathrm{U}(N)} = \exp\left(\sum_{n=1}^\infty \frac{1}{n} F_N(q_1^n, q_2^n, q_3^n; \Lambda^n)\right),\tag{4.1}$$

where

$$F_N := -\frac{\widetilde{\Lambda}(1 + \mathbf{q}^2 + \mathbf{q}^4 + \cdots + \mathbf{q}^{2N-2})}{(1 - \widetilde{\Lambda})(1 - \mathbf{q}^{2N}\widetilde{\Lambda})} F_0(q_1, q_2, q_3),\tag{4.2}$$

and $\widetilde{\Lambda} = (-\mathbf{q})^{-N}\Lambda$. For later convenience we have introduced

$$F_0(q_1, q_2, q_3) := \frac{(1 - q_1 q_2)(1 - q_2 q_3)(1 - q_3 q_1)}{(1 - q_1)(1 - q_2)(1 - q_3)}.\tag{4.3}$$

For U(1) theory the same conjecture was already given by Nekrasov [24] and the above proposal is a generalization to U(N) theory. It may look that there is only a little difference between U(N) and U(1) theories. However, we would like to emphasize that it is a consequence of the crucial fact that the partition function does not depend on the equivariant parameters from the maximal torus U(1)^N, or the vacuum expectation values of Higgs scalars. If we impose the Calabi-Yau condition $\mathbf{q} = 1$, our conjecture implies

$$F_N = \frac{Nu}{(1 - u)^2},\tag{4.4}$$

with $\widetilde{\Lambda} = (-1)^N\Lambda = u$. Thus we recover the result of the last section. In this sense the above instanton partition function suggests a generalization of Donaldson-Thomas theory to Kähler manifold.

Let us consider the following instanton expansion

$$Z_{6D}^{U(N)} = 1 + \sum_{k=1}^{\infty} \widetilde{\Lambda}^k Z_N^{(k)}(q_1, q_2, q_3),$$

(4.5)

$$F_N = 1 + \sum_{k=1}^{\infty} \widetilde{\Lambda}^k F_N^{(k)}(q_1, q_2, q_3).$$

(4.6)

It is quite amusing that since

$$-\frac{\widetilde{\Lambda}(1 + \mathbf{q}^2 + \mathbf{q}^4 + \cdots + \mathbf{q}^{2N-2})}{(1 - \widetilde{\Lambda})(1 - \mathbf{q}^{2N}\widetilde{\Lambda})} = \frac{-1}{1 - \mathbf{q}^2}\left(\frac{1}{1 - \widetilde{\Lambda}} - \frac{1}{1 - \mathbf{q}^{2N}\widetilde{\Lambda}}\right) = -\sum_{k=1}^{\infty}\frac{1 - \mathbf{q}^{2Nk}}{1 - \mathbf{q}^2}\widetilde{\Lambda}^k,$$

(4.7)

the coefficients of the instanton expansion of F_N take a very simple form;

$$F_N^{(k)}(q_1, q_2, q_3) = -\mathbf{q}^{Nk-1}[Nk]_{\mathbf{q}} \cdot F_0(q_1, q_2, q_3),$$

(4.8)

where the $\mathbf{q}$-integer is defined by

$$[n]_{\mathbf{q}} := \frac{\mathbf{q}^n - \mathbf{q}^{-n}}{\mathbf{q} - \mathbf{q}^{-1}} = \mathbf{q}^{1-n}\frac{1 - \mathbf{q}^{2n}}{1 - \mathbf{q}^2}.$$

(4.9)

We have the $\mathbf{q}$-binomial theorem ([41]; Chap.I-2,Example 3), which is useful in the following computation,

$$\exp\left\{-\sum_{n>0}\frac{(-z)^n}{n}[N]_{\mathbf{q}^n}\right\} = \prod_{\alpha=1}^{N}(1 + z\mathbf{q}^{N+1-2\alpha}) = \sum_{k=0}^{N}z^k\begin{bmatrix}N\\k\end{bmatrix}_{\mathbf{q}},$$

(4.10)

with

$$\begin{bmatrix}N\\k\end{bmatrix}_{\mathbf{q}} := \frac{[N]!_{\mathbf{q}}}{[N-k]!_{\mathbf{q}}[k]!_{\mathbf{q}}}, \qquad [N]!_{\mathbf{q}} := [N]_{\mathbf{q}}[N-1]_{\mathbf{q}}\cdots[1]_{\mathbf{q}}.$$

(4.11)

From this we obtain

$$\mathbf{q}^{k(N+1)}\sum_{1\le\alpha_i<\alpha_j\le N}\prod_{i=1}^{k}\mathbf{q}^{-2\alpha_i} = \begin{bmatrix}N\\k\end{bmatrix}_{\mathbf{q}},$$

(4.12)

and

$$[N]_{\mathbf{q}^2} = \begin{bmatrix}N\\1\end{bmatrix}_{\mathbf{q}}^2 - 2\begin{bmatrix}N\\2\end{bmatrix}_{\mathbf{q}},$$

(4.13)

$$[N]_{\mathbf{q}^3} = \begin{bmatrix}N\\1\end{bmatrix}_{\mathbf{q}}^3 - 3\begin{bmatrix}N\\1\end{bmatrix}_{\mathbf{q}}\begin{bmatrix}N\\2\end{bmatrix}_{\mathbf{q}} + 3\begin{bmatrix}N\\3\end{bmatrix}_{\mathbf{q}}.$$

(4.14)

In terms of $F_N^{(k)}$, the instanton expansion of the partition function is

$$Z_N^{(1)}(q_1, q_2, q_3) = F_N^{(1)}(q_1, q_2, q_3),$$

$$Z_N^{(2)}(q_1, q_2, q_3) = F_N^{(2)}(q_1, q_2, q_3) + \frac{1}{2}\left(F_N^{(1)}(q_1, q_2, q_3)\right)^2 + \frac{1}{2}F_N^{(1)}(q_1^2, q_2^2, q_3^2),$$

(4.15)

$$Z_N^{(3)}(q_1, q_2, q_3) = F_N^{(3)}(q_1, q_2, q_3) + F_N^{(2)}(q_1, q_2, q_3)F_N^{(1)}(q_1, q_2, q_3)$$

$$+ \frac{1}{2}F_N^{(1)}(q_1^2, q_2^2, q_3^2)F_N^{(1)}(q_1, q_2, q_3) + \frac{1}{3}F_N^{(1)}(q_1^3, q_2^3, q_3^3) + \frac{1}{6}\left(F_N^{(1)}(q_1, q_2, q_3)\right)^3.$$

In the following subsections we prove the conjecture up to three instanton number for any N.

4.1 One instanton

The fixed points with $k = 1$ are the colored plane partition $(\Box, \bullet, \cdots, \bullet)$ and its cyclic permutations, where $\Box$ stands for the plane partition with unit volume. The character of the fixed point $\vec{\pi}(\alpha)$ with $V^*_{\vec{\pi}(\alpha)} = e_\alpha$ is

$$\chi_{\vec{\pi}(\alpha)} = \sum_{\beta \neq \alpha} e_\alpha e_\beta^{-1} - \mathbf{q}^2 \sum_{\beta \neq \alpha} e_\beta e_\alpha^{-1} + (q_1 + q_2 + q_3 - q_1 q_2 - q_2 q_3 - q_3 q_1), \tag{4.16}$$

and

$$N_{\vec{\pi}(\alpha)}(q_\ell, e_\lambda) = \mathbf{q}^{-N} \frac{(1 - q_1 q_2)(1 - q_2 q_3)(1 - q_3 q_1)}{(1 - q_1)(1 - q_2)(1 - q_3)} \prod_{\beta \neq \alpha} \frac{e_\alpha - \mathbf{q}^2 e_\beta}{e_\beta - e_\alpha} . \tag{4.17}$$

We can show that

$$\sum_{\alpha=1}^{N} \prod_{\beta \neq \alpha} \frac{e_\alpha - \mathbf{q}^2 e_\beta}{e_\beta - e_\alpha} = (-1)^{N-1} \left(1 + \mathbf{q}^2 + \mathbf{q}^4 + \cdots + \mathbf{q}^{2N-2}\right) . \tag{4.18}$$

In fact possible poles in the left hand side are at $e_\alpha = e_\beta$. But we see that

$$\operatorname{Res}_{e_\alpha = e_\beta} N_{\vec{\pi}(\alpha)} = -\operatorname{Res}_{e_\alpha = e_\beta} N_{\vec{\pi}(\beta)} . \tag{4.19}$$

Hence all the residues vanish and the left hand side is a constant in e_α. We may compute it by putting $e_\alpha = L^{-\alpha}$, $(1 \leq \alpha \leq N)$ and taking the limit $L \to \infty$ to obtain (4.18). Thus we find that $Z_N^{(1)}$ does not depend on e_α, which physically means it is independent of a_α, or the relative distances of N $D6$ branes. The partition function at one instanton is

$$Z_N^{(1)} = \sum_{\alpha=1}^{N} N_{\vec{\pi}(\alpha)} = (-\mathbf{q})^{-1}(-1)^N [N]_{\mathbf{q}} \cdot F_0(q_1, q_2, q_3), \tag{4.20}$$

which proves the conjecture at one instanton.

4.2 Two instanton

Two instanton part of the partition function is computed as follows; we have two types of configuration, whose characters are $V^*_{\vec{\pi}(\alpha,i)} := e_\alpha(1 + q_i), 1 \leq \alpha \leq N, i = 1, 2, 3$, which we call type I in the following and $V^*_{\vec{\pi}(\alpha,\beta)} := e_\alpha + e_\beta, 1 \leq \alpha < \beta \leq N$, which we call type II.

For type I we find

$$N_{\vec{\pi}(\alpha,i)}(q_\ell, e_\lambda) = \mathbf{q}^{-2N} n_{\mathrm{I}}^{(i)}(q_\ell) \prod_{\beta \neq \alpha} \frac{(e_\beta - e_\alpha \mathbf{q}^2)(e_\beta - q_i e_\alpha \mathbf{q}^2)}{(e_\alpha - e_\beta)(q_i e_\alpha - e_\beta)}, \tag{4.21}$$

where

$$n_{\mathrm{I}}^{(i)}(q_\ell) := \frac{(q_i - \prod_{j \neq i} q_j) \prod_{j \neq i}(1 - q_i^2 q_j)}{(1 - q_i^2) \prod_{j \neq i}(q_i - q_j)} F_0(q_1, q_2, q_3) . \tag{4.22}$$

Similarly for the second type we have

$$N_{\vec{\pi}(\alpha,\beta)}(q_\ell, e_\lambda) = \mathbf{q}^{-2N} n_{\mathrm{II}}(q_\ell) \frac{\prod_{1 \leq i < j \leq 3}(e_\alpha - e_\beta q_i q_j)(e_\beta - e_\alpha q_i q_j)}{\prod_{i=1}^{3}(e_\alpha - e_\beta q_i)(e_\beta - e_\alpha q_i)} \prod_{\gamma \neq \alpha, \beta} \frac{(e_\gamma - e_\alpha \mathbf{q}^2)(e_\gamma - e_\beta \mathbf{q}^2)}{(e_\alpha - e_\gamma)(e_\beta - e_\gamma)}, \tag{4.23}$$

where
$$n_{\mathrm{II}}(q_\ell) := F_0(q_1, q_2, q_3)^2 \,. \tag{4.24}$$

Let us look at possible poles and residues there. There are poles at $e_\alpha = e_\beta$ and $q_i e_\alpha = e_\beta$. Taking the relation $\mathbf{q}^2 = q_1 q_2 q_3$ into account, we see the relations

$$
\begin{aligned}
\mathrm{Res}_{e_\alpha = e_\beta} \left(N_{\vec{\pi}(\alpha, i)} + N_{\vec{\pi}(\beta, i)} \right) &= 0, \qquad i = 1, 2, 3, \\
\mathrm{Res}_{e_\alpha = e_\beta} \left(N_{\vec{\pi}(\alpha, \gamma)} + N_{\vec{\pi}(\beta, \gamma)} \right) &= 0, \qquad 1 \leq \gamma \leq N, \quad \gamma \neq \alpha, \beta, \\
\mathrm{Res}_{q_i e_\alpha = e_\beta} \left(N_{\vec{\pi}(\alpha, i)} + N_{\vec{\pi}(\alpha, \beta)} \right) &= 0.
\end{aligned}
\tag{4.25}
$$

Therefore, the partition function does not depend on e_α. By estimating the leading terms $e_\alpha = L^{-\alpha}$, $L \to \infty$, we find the two instanton part of the partition function is

$$
\begin{aligned}
Z_N^{(2)} &= \mathbf{q}^{-2N} \left(\sum_{\alpha=1}^{N} \mathbf{q}^{4\alpha-4} \sum_{i=1}^{3} n_{\mathrm{I}}^{(i)}(q_\ell) + \sum_{1 \leq \alpha < \beta \leq N} \mathbf{q}^{2\alpha+2\beta-4} n_{\mathrm{II}}(q_\ell) \right) \\
&= \mathbf{q}^{-2} \left([N]_{\mathbf{q}^2} \sum_{i=1}^{3} n_{\mathrm{I}}^{(i)}(q_\ell) + \begin{bmatrix} N \\ 2 \end{bmatrix}_{\mathbf{q}} n_{\mathrm{II}}(q_\ell) \right) \,.
\end{aligned}
\tag{4.26}
$$

On the other hand the conjecture says

$$
\begin{aligned}
Z_N^{(2)} = \mathbf{q}^{-2} \Big(&-(1 + \mathbf{q}^2)[N]_{\mathbf{q}^2} \cdot F_0(q_1, q_2, q_3) + \frac{1}{2}[N]_{\mathbf{q}}^2 \cdot F_0(q_1, q_2, q_3)^2 \\
&-\frac{1}{2}[N]_{\mathbf{q}^2} \frac{(1 + q_1 q_2)(1 + q_2 q_3)(1 + q_3 q_1)}{(1 + q_1)(1 + q_2)(1 + q_3)} F_0(q_1, q_2, q_3) \Big) \,.
\end{aligned}
\tag{4.27}
$$

Using the identity (4.13) we can see that the conjecture at two instanton reduces to the following identity;
$$[N]_{\mathbf{q}^2} \cdot F_0(q_1, q_2, q_3) \cdot G_{\mathrm{U}(1)}(q_1, q_2, q_3) = 0 \,, \tag{4.28}$$
where $G_{\mathrm{U}(1)} = 0$ is equivalent to the identity

$$
\begin{aligned}
&\frac{(q_1 - q_2 q_3)(1 - q_1^2 q_2)(1 - q_1^2 q_3)}{(1 - q_1^2)(q_1 - q_2)(q_1 - q_3)} + (1, 2, 3) \text{ cyclic} \\
&= -(1 + \mathbf{q}^2) - \frac{1}{2} \frac{(1 + q_1 q_2)(1 + q_2 q_3)(1 + q_3 q_1)}{(1 + q_1)(1 + q_2)(1 + q_3)} + \frac{1}{2} \frac{(1 - q_1 q_2)(1 - q_2 q_3)(1 - q_3 q_1)}{(1 - q_1)(1 - q_2)(1 - q_3)} \,.
\end{aligned}
\tag{4.29}
$$

The crucial point is that N dependence is factored out and the remaining factor $G_{\mathrm{U}(1)}$ is universal in the sense that it is independent of the rank N. That is what we have to prove for general N is the same as that for U(1) case. Actually the identity (4.29) is necessary for proving the conjecture for U(1) theory. In this case type II configuration does not appear and the proof of the conjecture is easier. We note that the identity (4.29) is transformed into the following form

$$\sum_{i=1}^{3} \frac{p - q_i^2}{1 - q_i^2} \prod_{j(\neq i)} \frac{p q_i - q_j}{q_i - q_j} = p(1 + p) + \frac{1}{2} \prod_{\ell=1}^{3} \frac{p - q_\ell}{1 - q_\ell} + \frac{1}{2} \prod_{i=1}^{3} \frac{p + q_\ell}{1 + q_\ell} \,, \tag{4.30}$$

if $p = q_1 q_2 q_3$. Hence one can derive (4.29) from the partial fraction decomposition

$$\prod_{\ell=1}^{n} \frac{pz - x_\ell}{z - x_\ell} = \sum_{i=1}^{n} \frac{p - x_i}{z - x_i} \prod_{j(\neq i)} \frac{px_i - x_j}{x_i - x_j} - \sum_{i=1}^{n-1} p^i, \qquad (4.31)$$

with $n = 3$ and $x_i = \pm z q_i$.

The fact that the proof is essentially reduced to abelian case might be expected. We know that the result is independent of the vacuum expectation values of Higgs fields by looking at residues. This means the partition function does not depend on relative distances of $D6$ branes and hence we can compute it by taking the decoupling limit where $D6$ branes are infinitely separated. In fact the leading term mentioned above can be regarded as the result in this limit.

4.3 Three instanton

We have four types of configurations;

1. Type A_1 $\quad V^*_{\vec{\pi}(\alpha,i)} = e_\alpha(1 + q_i + q_i^2), \quad 1 \leq \alpha \leq N, \quad i = 1, 2, 3$

$$N_{\vec{\pi}(\alpha,i)}(q_\ell, e_\lambda) = \mathbf{q}^{-3N} n_{A_1}^{(i)}(q_\ell) \prod_{\beta \neq \alpha} \frac{(e_\beta - e_\alpha \mathbf{q}^2)(e_\beta - q_i e_\alpha \mathbf{q}^2)(e_\beta - q_i^2 e_\alpha \mathbf{q}^2)}{(e_\alpha - e_\beta)(q_i e_\alpha - e_\beta)(q_i^2 e_\alpha - e_\beta)}, \quad (4.32)$$

where

$$n_{A_1}^{(i)}(q_\ell) := \frac{(q_i - \prod_{j \neq i} q_j)(q_i^2 - \prod_{j \neq i} q_j) \prod_{j \neq i}(1 - q_i^2 q_j)(1 - q_i^3 q_j)}{(1 - q_i^2)(1 - q_i^3) \prod_{j \neq i}(q_i - q_j)(q_i^2 - q_j)} F_0(q_1, q_2, q_3) .$$
$$(4.33)$$

2. Type A_2 $\quad V^*_{\vec{\pi}(\alpha,i,j)} = e_\alpha(1 + q_i + q_j), \quad 1 \leq \alpha \leq N, \quad 1 \leq i < j \leq 3$

$$N_{\vec{\pi}(\alpha,i,j)}(q_\ell, e_\lambda) = \mathbf{q}^{-3N} n_{A_2}^{(i,j)}(q_\ell) \prod_{\beta \neq \alpha} \frac{(e_\beta - e_\alpha \mathbf{q}^2)(e_\beta - q_i e_\alpha \mathbf{q}^2)(e_\beta - q_j e_\alpha \mathbf{q}^2)}{(e_\alpha - e_\beta)(q_i e_\alpha - e_\beta)(q_j e_\alpha - e_\beta)}, \quad (4.34)$$

where with $k \neq i, j$

$$n_{A_2}^{(i,j)}(q_\ell) := \frac{(1 - q_i q_k)(1 - q_j q_k)(1 - q_i^2 q_j)(1 - q_i q_j^2)(q_i - q_j^2 q_k)(q_j - q_i^2 q_k)}{(1 - q_i)(1 - q_j)(q_i - q_k)(q_j - q_k)(q_i - q_j^2)(q_j - q_i^2)} F_0(q_1, q_2, q_3) .$$
$$(4.35)$$

3. Type B $\quad V^*_{\vec{\pi}(\alpha,\beta,i)} = e_\alpha(1 + q_i) + e_\beta, \quad 1 \leq \alpha \neq \beta \leq N, \quad i = 1, 2, 3$

$$N_{\vec{\pi}(\alpha,\beta,i)}(q_\ell, e_\lambda) = \mathbf{q}^{-3N} n_B^{(i)}(q_\ell) \prod_{\gamma \neq \alpha, \beta} \frac{(e_\gamma - e_\alpha \mathbf{q}^2)(e_\gamma - e_\beta \mathbf{q}^2)(e_\gamma - q_i e_\alpha \mathbf{q}^2)}{(e_\alpha - e_\gamma)(e_\beta - e_\gamma)(q_i e_\alpha - e_\gamma)}$$

$$\times \frac{(e_\alpha q_i - e_\beta \prod_{j \neq i} q_j)(e_\beta - e_\alpha \mathbf{q}^2)(e_\beta - e_\alpha \prod_{j \neq i} q_j) \prod_{j \neq i}(e_\beta - e_\alpha q_i^2 q_j)(e_\alpha - e_\beta q_i q_j)}{(e_\alpha - e_\beta)(e_\alpha - e_\beta q_i)(e_\beta - e_\alpha q_i^2) \prod_{j \neq i}(e_\alpha q_i - e_\beta q_j)(e_\beta - e_\alpha q_j)},$$
$$(4.36)$$

where

$$n_B^{(i)}(q_\ell) := \frac{(q_i - \prod_{j \neq i} q_j) \prod_{j \neq i}(1 - q_i^2 q_j)}{(1 - q_i^2) \prod_{j \neq i}(q_i - q_j)} F_0(q_1, q_2, q_3)^2 = n_{\mathrm{I}}^{(i)}(q_\ell) F_0(q_1, q_2, q_3) \ .$$

(4.37)

4. Type C $V_{\vec{\pi}(\alpha,\beta,\gamma)}^* = e_\alpha + e_\beta + e_\gamma, \quad 1 \leq \alpha < \beta < \gamma \leq N$

$$N_{\vec{\pi}(\alpha,\beta,\gamma)}(q_\ell, e_\lambda) = \mathbf{q}^{-3N} n_C(q_\ell) \prod_{a,b=\alpha,\beta,\gamma} \frac{\prod_{1 \leq i < j \leq 3}(e_a - e_b q_i q_j)}{\prod_{i=1}^3 (e_a - e_b q_i)}$$
$$\times \prod_{\delta \neq \alpha,\beta,\gamma} \frac{(e_\delta - e_\alpha \mathbf{q}^2)(e_\delta - e_\beta \mathbf{q}^2)(e_\delta - e_\gamma \mathbf{q}^2)}{(e_\alpha - e_\delta)(e_\beta - e_\delta)(e_\gamma - e_\delta)} \ , \quad (4.38)$$

where

$$n_C(q_\ell) := F_0(q_1, q_2, q_3)^3 = n_{\mathrm{II}}(q_\ell) F_0(q_1, q_2, q_3) \ .$$

(4.39)

As before all residues cancel out between two terms as follows;

$$\mathrm{Res}_{e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,i)} + N_{\vec{\pi}(\beta,i)}\right) = 0, \qquad \mathrm{Res}_{q_i e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,i)} + N_{\vec{\pi}(\beta,\alpha,i)}\right) = 0,$$
$$\mathrm{Res}_{e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,i,j)} + N_{\vec{\pi}(\beta,i,j)}\right) = 0, \qquad \mathrm{Res}_{q_i e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,i,j)} + N_{\vec{\pi}(\alpha,\beta,j)}\right) = 0,$$
$$\mathrm{Res}_{e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,\beta,i)} + N_{\vec{\pi}(\beta,\alpha,i)}\right) = 0, \qquad \mathrm{Res}_{q_i e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,\gamma,i)} + N_{\vec{\pi}(\alpha,\beta,\gamma)}\right) = 0, \quad (4.40)$$
$$\mathrm{Res}_{q_i^2 e_\alpha = e_\beta}\left(N_{\vec{\pi}(\alpha,i)} + N_{\vec{\pi}(\alpha,\beta,i)}\right) = 0, \quad \mathrm{Res}_{q_i e_\alpha = q_j e_\beta}\left(N_{\vec{\pi}(\alpha,\beta,i)} + N_{\vec{\pi}(\beta,\alpha,j)}\right) = 0,$$

with $\gamma \neq \alpha, \beta$ and $j \neq i$. Thus we can confirm that the partition function does not depend on e_α and compute the partition function by taking the decoupling limit as before. The three instanton part of the partition function is

$$Z_N^{(3)} = \mathbf{q}^{-3N}(-1)^{N-1}\left(\sum_{\alpha=1}^N \mathbf{q}^{6\alpha-6} \sum_{i=1}^3 n_{A_1}^{(i)}(q_\ell) + \sum_{\alpha=1}^N \mathbf{q}^{6\alpha-6} \sum_{(i,j)} n_{A_2}^{(i,j)}(q_\ell) \right.$$
$$\left. + \sum_{1 \leq \alpha \neq \beta \leq N} \mathbf{q}^{4\alpha+2\beta-6} \sum_{i=1}^3 n_B^{(i)}(q_\ell) + \sum_{1 \leq \alpha < \beta < \gamma \leq N} \mathbf{q}^{2\alpha+2\beta+2\gamma-6} \cdot n_C(q_\ell)\right)$$
$$= \mathbf{q}^{-3}(-1)^{N-1}\left([N]_{\mathbf{q}^3} \sum_{i=1}^3 n_{A_1}^{(i)}(q_\ell) + [N]_{\mathbf{q}^3} \sum_{(i,j)} n_{A_2}^{(i,j)}(q_\ell) \right.$$
$$\left. + ([N]_{\mathbf{q}^2}[N]_{\mathbf{q}} - [N]_{\mathbf{q}^3}) \sum_{i=1}^3 n_B^{(i)}(q_\ell) + \begin{bmatrix} N \\ 3 \end{bmatrix}_{\mathbf{q}} n_C(q_\ell)\right) \ .$$

(4.41)

The conjecture implies

$$
\begin{aligned}
Z_N^{(3)} = \mathbf{q}^{-3} \Big(&- (1 + \mathbf{q}^2 + \mathbf{q}^4)[N]_{\mathbf{q}^3} \cdot F_0(q_1, q_2, q_3) + (1 + \mathbf{q}^2)[N]_{\mathbf{q}^2}[N]_{\mathbf{q}} \cdot F_0(q_1, q_2, q_3)^2 \\
&+ \frac{1}{2}[N]_{\mathbf{q}^2}[N]_{\mathbf{q}} \frac{(1 + q_1 q_2)(1 + q_2 q_3)(1 + q_3 q_1)}{(1 + q_1)(1 + q_2)(1 + q_3)} F_0(q_1, q_2, q_3)^2 \\
&- \frac{1}{3}[N]_{\mathbf{q}^3} \frac{(1 + q_1 q_2 + q_1^2 q_2^2)(1 + q_2 q_3 + q_2^2 q_3^2)(1 + q_3 q_1 + q_3^2 q_1^2)}{(1 + q_1 + q_1^2)(1 + q_2 + q_2^2)(1 + q_3 + q_3^2)} F_0(q_1, q_2, q_3) \\
&- \frac{1}{6}[N]_{\mathbf{q}}^3 F_0(q_1, q_2, q_3)^3 \Big) \ .
\end{aligned}
\tag{4.42}
$$

Using (4.14) and (4.29) which we have used at two instanton, we see that the conjecture boils down to

$$
[N]_{\mathbf{q}^3} \cdot F_0(q_1, q_2, q_3) \cdot H_{\mathrm{U}(1)}(q_1, q_2, q_3) = 0 \,,
\tag{4.43}
$$

where $H_{\mathrm{U}(1)}(q_1, q_2, q_3) = 0$ is equivalent to the identity

$$
\begin{aligned}
\sum_{i=1}^{3} \prod_{n=1}^{2} \frac{p - q_i^{n+1}}{1 - q_i^{n+1}} \prod_{j(\neq i)} \frac{p q_i^n - q_j}{q_i^n - q_j} &+ \sum_{\substack{i<j \\ k\neq i,j}} \frac{p q_i - q_k}{q_i - q_k} \frac{p q_j - q_k}{q_j - q_k} \prod_{n=1}^{2} \frac{p q_i^{n-1} - q_j^n}{q_i^{n-1} - q_j^n} \frac{p q_j^{n-1} - q_i^n}{q_j^{n-1} - q_i^n} \\
&= p^2(1 + p + p^2) + p(1 + p)f(p, q_\ell) + \frac{1}{2}f(p^2, q_\ell^2) + \frac{1}{3}\frac{f(p^3, q_\ell^3)}{f(p, q_\ell)} + \frac{1}{3!}f(p, q_\ell)^2 \,, \ (4.44)
\end{aligned}
$$

with $f(p, q_\ell) := \prod_{\ell=1}^{3}(p - q_\ell)/(1 - q_\ell)$, if $p = q_1 q_2 q_3$. Again we can factor out N dependence completely and what we have to show is the identity (4.44), which is required for proving the conjecture for U(1) theory. Note that in U(1) case the colored plane partitions of type B and C do not appear. We can check the identity (4.44) by direct computation based on the partial fraction decomposition.

In summary, computations of instanton number two and three show that basic ingredients for the validity of the conjecture are identities for $\mathbf{q}$-integers such as (4.13) and (4.14) and the combinatorial identity for U(1) theory like (4.29) and (4.44). We believe we will see similar structure for higher instanton numbers. In fact (4.13) and (4.14) are the first two identities which are derived form the $\mathbf{q}$-binomial theorem (4.10). On the other hand at the moment we cannot see any underlying reason for the identities (4.29) and (4.44), though we can check them by considering the partial fraction decomposition. Since they are the equalities for U(1) theory, it is tempting to expect that they are related to the geometry or combinatorics of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^3)$ of points in $\mathbb{C}^3$.

Acknowledgments

A ADHM matrix model and Nekrasov's partition function

In this appendix we review how we can derive Nekrasov's instanton partition function as an equivariant index of the matrix quantum mechanics of the ADHM equations. Let us consider two vector spaces V and W with complex dimensions, $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} W = N$. In the language of D brane system we have k $D0$ branes bound to N $D4$ branes. As an effective theory on $D4$ branes we have $U(N)$ gauge theory and k $D0$ branes describe the gas of point-like k instantons. In the D brane picture the ADHM construction is a dual description where we consider an effective $0+1$ dimensional theory on $D0$ brane [19–22]. We have $B_{1,2} \in \mathrm{Hom}\,(V, V)$ from "0-0" string. From "0-4" and "4-0" string we have $I \in \mathrm{Hom}\,(W, V)$ and $J \in \mathrm{Hom}\,(V, W)$. The ADHM equations for these ADHM data are

$$\mathcal{E}_{\mathbb{C}} := [B_1, B_2] + IJ = 0, \tag{A.1}$$

$$\mathcal{E}_{\mathbb{R}}(\zeta) := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J - \zeta = 0 . \tag{A.2}$$

When we construct the moduli space of instantons as the hyperKähler quotient, they play the role of hyperKähler moment maps. Namely the moduli space can be identified with

$$\mathcal{M}_{\mathrm{ADHM}} := \{(B_1, B_2, I, J)|\ \mathcal{E}_{\mathbb{C}} = 0, \mathcal{E}_{\mathbb{R}}(\zeta) = 0\}/U(k) . \tag{A.3}$$

The formal dimension of $\mathcal{M}_{\mathrm{ADHM}}$ is computed as follows; we impose $2k^2 + k^2$ (real) constraints on $4k^2 + 4NK$ (real) degrees of freedom from the matrices (B_1, B_2, I, J). Since the gauge group $U(k)$ reduces further k^2 degrees of freedom, we find the moduli space has $4NK$ dimensions, or $\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{ADHM}} = 2Nk$, which agrees to the dimensions of the moduli space of ASD instanton of $U(N)$ theory with instanton number k. It is known that the moduli space is isomorphic to the following affine algebro-geometric quotient [3, 23];

$$\widetilde{\mathcal{M}}_{\mathrm{ADHM}} := \{(B_1, B_2, I, J)|\ \mathcal{E}_{\mathbb{C}} = 0\}//GL(k, \mathbb{C}) . \tag{A.4}$$

In (A.4) instead of the D term condition we impose the algebraic stability condition that there is no proper subspace S of V which satisfies $B_1 S \subset S, B_2 S \subset S$ and $\mathrm{Im}\,(I) \subset S$.

We consider the toric action $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$ of T^2 on $\mathbb{C}^2$. The ADHM data transform $(B_1, B_2, I, J) \to (q_1 \cdot B_1, q_2 \cdot B_2, I, (q_1 q_2) \cdot J)$ where $q_i := e^{i\epsilon_i}$. The fixed points are isolated and classified by N-tuples of partitions $\vec{\lambda}$ [23]. The equivariant deformation complex at a fixed point $\vec{\lambda}$ is [3, 23, 42–44];

$$\begin{array}{ccccc}
 & & \mathrm{Hom}\,(V_{\vec{\lambda}}, V_{\vec{\lambda}}) \otimes Q & & \\
 & & \oplus & & \\
\mathrm{Hom}\,(V_{\vec{\lambda}}, V_{\vec{\lambda}}) & \xrightarrow{\sigma} & \mathrm{Hom}\,(W_{\vec{\lambda}}, V_{\vec{\lambda}}) & \xrightarrow{\tau} & \mathrm{Hom}\,(V_{\vec{\lambda}}, V_{\vec{\lambda}}) \otimes \Lambda^2 Q, \\
 & & \oplus & & \\
 & & \mathrm{Hom}\,(V_{\vec{\lambda}}, W_{\vec{\lambda}}) \otimes \Lambda^2 Q & &
\end{array} \tag{A.5}$$

where $Q = T_1^{-1} + T_2^{-1}$ and T_i is one dimensional module on which T^2 acts as the multiplication of $e^{i\epsilon_i}$. Hence the equivariant index is

$$\chi = (V^* \otimes V)Q + W^* \otimes V + V^* \otimes W \otimes \Lambda^2 Q - (V^* \otimes V)(1 + \Lambda^2 Q)$$
$$= W^* \otimes V + V^* \otimes W(T_1 T_2)^{-1} - V^* \otimes V(1 - T_1^{-1})(1 - T_2^{-1}) . \tag{A.6}$$

We have $2NK$ positive terms in this index which are regarded as the weights (eigenvalues) of the toric action at the fixed points.[5] Each weight is a monomial in the equivariant parameters $q_i^{\pm} = e^{\pm i\epsilon_i}$ from T^2 and $e_\alpha^{\pm} = e^{\pm ia_\alpha}$. Hence from a character of the form $\chi = \sum_{i=1}^{2Nk} \exp(w_i)$, we obtain the following contribution to the instanton partition function;

$$z(\vec{\lambda}) = \prod_{i=1}^{2Nk} (1 - \exp(w_i))^{-1} , \tag{A.7}$$

where we consider the K theoretic version of the partition function, which corresponds to the index of the Dolbeault operator $\bar{\partial}$ or the Todd class. By localization theorem the partition function is computed by summing up all the contributions at each fixed point, or the colored partition $\vec{\lambda}$;

$$Z_{\text{Nek}}(e_\alpha, q_i; \Lambda) = \sum_{\vec{\lambda}} \left(\frac{\Lambda}{\sqrt{q_1 q_2}} \right)^{N|\vec{\lambda}|} \frac{1}{\prod_{\alpha,\beta=1}^{N} N_{\alpha,\beta}(e_\alpha, q_i)} , \tag{A.8}$$

where

$$N_{\alpha,\beta}(e_\alpha, q_i) = \prod_{s\in\lambda_\alpha} \left(1 - q_1^{-\ell_{\lambda_\beta}(s)-1} q_2^{a_{\lambda_\alpha}(s)} e_\alpha e_\beta^{-1} \right) \prod_{t\in\lambda_\beta} \left(1 - q_1^{\ell_{\lambda_\alpha}(t)} q_2^{-a_{\lambda_\beta}(t)-1} e_\alpha e_\beta^{-1} \right) . \tag{A.9}$$

Note that we have renormalized the parameter Λ of instanton expansion by $\sqrt{q_1 q_2}$ as we made for the topological partition function in this paper.

B Well-definedness of $N_{\vec{\pi}}(q_i, e_\alpha)$

To define the weight function $N_{\vec{\pi}}(q_i, e_\alpha)$, $\{e^{w_i^{(+)}}\}$, which is defined by (3.12), should not contain 1. We prove it here.

A plane partition π is define as a finite set of positive integers, $\pi = \{(i,j,k)\} \subset \mathbb{N}^3$, such that if $(i,j,k) \in \pi$ then $(i',j',k') \in \pi$ ($1 \le i' \le i$, $1 \le j' \le j$, $1 \le k' \le k$). Given any plane partition π, let

$$n(s;t) := \# \left\{ (i,j,k) \in \pi \,|\, (i',j',k') := (i-s_1-t_1, j-s_2-t_2, k-s_3-t_3) \in \pi \right\}, \tag{B.1}$$

with $s = (s_1, s_2, s_3)$, $t = (t_1, t_2, t_3)$ and

$$
\begin{aligned}
n_0(s) &:= n(s;0,0,0), & n_{-1}(s) &:= \# \{(s_1+1, s_2+1, s_3+1) \in \pi\}, \\
n_1(s) &:= n(s;1,0,0) + n(s;0,1,0) + n(s;0,0,1), \\
n_2(s) &:= n(s;0,1,1) + n(s;1,0,1) + n(s;1,1,0), \\
n_3(s) &:= n(s;1,1,1). & &\tag{B.2}
\end{aligned}
$$

Note that

$$n_0(0,0,0) = |\pi|, \qquad n_{-1}(0,0,0) = \begin{cases} 0, & \pi = \emptyset \\ 1, & \pi \ne \emptyset \end{cases} . \tag{B.3}$$

---

[5]In the character (A.6) all the term with negative coefficient are canceled and there are $2Nk$ remaining terms.

First we have

Lemma. If $(s_1, s_2, s_3) \in \mathbb{Z}_{\geq 0}^3$ then $\sum_{\ell=-1}^{3} (-1)^\ell n_\ell(s) = 0$.

Proof. For $\pi = \emptyset$, since $n_\ell(s) = 0$, the lemma holds. Assuming the lemma to hold for π, we will prove it for the plane partition $\pi' = \pi \cup \{(i,j,k)\}$. The differences between $n_\ell(s)$'s of π and those of π', $(\Delta n_{-1}(s), \Delta n_0(s), \Delta n_1(s), \Delta n_2(s), \Delta n_3(s))$, are

$$
\begin{aligned}
(0,1,3,3,1), &\qquad i - s_1, j - s_2, k - s_3 > 1, \\
(0,1,2,1,0), &\qquad \{i - s_1, j - s_2, k - s_3\} = \{1, \alpha, \beta\}, \\
(0,1,1,0,0), &\qquad \{i - s_1, j - s_2, k - s_3\} = \{1, 1, \alpha\}, \\
(1,1,0,0,0), &\qquad i - s_1 = j - s_2 = k - s_3 = 1, \\
(0,0,0,0,0), &\qquad i - s_1 \quad \text{or} \quad j - s_2 \quad \text{or} \quad k - s_3 < 1,
\end{aligned}
\tag{B.4}
$$

with $\alpha, \beta > 1$. Thus it holds for π'. $\qquad\square$

For $e^{w_i^{(\pm)}}$ introduced in (3.12) and (3.13), we have

Proposition. $e^{w_i^{(\pm)}} \neq q_1^{n_1} q_2^{n_2} q_3^{n_3}$ with $(n_1, n_2, n_3) \in \mathbb{Z}_{\leq 0}^3$ or $\in \mathbb{N}^3$.

Proof. It suffices to show it when $N = 1$, i.e., for

$$
\begin{aligned}
\chi_\pi(q_i, e_1) = \sum_{(i,j,k) \in \pi} q_1^{1-i} q_2^{1-j} q_3^{1-k} - \sum_{(i',j',k') \in \pi} q_1^{i'} q_2^{j'} q_3^{k'} \\
- \sum_{(i,j,k),(i',j',k') \in \pi} q_1^{i'-i} q_2^{j'-j} q_3^{k'-k} \left(1 - \sum_{\ell=1}^{3} q_\ell + \sum_{\ell=1}^{3} \frac{q_1 q_2 q_3}{q_\ell} - q_1 q_2 q_3 \right).
\end{aligned}
\tag{B.5}
$$

Each monomial $q_1^a q_2^b q_3^c$ in the 1st, 3rd, 4th, 5th and 6th terms of (B.5) becomes $q_1^{-s_1} q_2^{-s_2} q_3^{-s_3}$ ($s_i \in \mathbb{Z}_{\geq 0}$) if and only if $(i,j,k) = (s_1 + 1, s_2 + 1, s_3 + 1)$, $(i,j,k) - (i',j',k') = (s_1, s_2, s_3)$,

$$
\begin{aligned}
(i,j,k) - (i',j',k') - (s_1, s_2, s_3) &= (1,0,0) \quad \text{or} \quad (0,1,0) \quad \text{or} \quad (0,0,1), \\
(i,j,k) - (i',j',k') - (s_1, s_2, s_3) &= (0,1,1) \quad \text{or} \quad (1,0,1) \quad \text{or} \quad (1,1,0), \\
(i,j,k) - (i',j',k') - (s_1, s_2, s_3) &= (1,1,1),
\end{aligned}
\tag{B.6}
$$

respectively. But the number of them are $n_{-1}(s)$, $n_0(s)$, $n_1(s)$, $n_2(s)$ and $n_3(s)$, respectively, whose alternating summation vanishes. Thus $e^{w_i^{(\pm)}} \neq q_1^{n_1} q_2^{n_2} q_3^{n_3}$ with $(n_1, n_2, n_3) \in \mathbb{Z}_{\leq 0}^3$. The symmetry $\chi_{\vec{\pi}}(q_i, e_\alpha) = -q_1 q_2 q_3 \chi_{\vec{\pi}}(q_i^{-1}, e_\alpha^{-1})$ guarantees that $e^{w_i^{(\pm)}} \neq q_1^{n_1} q_2^{n_2} q_3^{n_3}$ with $(n_1, n_2, n_3) \in \mathbb{N}^3$. $\qquad\square$

Therefore, $N_{\vec{\pi}}(q_i, e_\alpha)$ is well-defined.

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